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SOME BASIC CONCEPTS FOR MAGNET COIL DESIGN

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SOME BASIC CONCEPTS FOR MAGNET COIL DESIGN

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In the United States and abroad there is intensive activity in thermonuclear research.[1]
A major part of this effort is directed toward containment, in strong magnetic fields, of
plasma of the necessary high particle energies, corresponding to temperatures of hundreds
of millions of degrees Kelvin. Magnetic field design in thermonuclear research has already
been developed to a relatively broad field of electrical engineering.[2] The purpose of this
paper is to discuss some theorems which are of special interest for the design of large d-c
magnet coils as used, for instance, in the DCX (Direct Current Experiment) thermonuclear
research program of the Oak Ridge National Laboratory.[3]

I. CYLINDRICAL MAGNET COILS WITH SQUARE ENDS AND UNIFORM CURRENT DENSITY

The design of high power magnet coils is based on fundamental work by Fabry,[4a,4b]
Cockcroft,[5] and Bitter.[6] It is advisable to summarize a few well-known points of this
theory. The simplest type of magnet coil is cylindrical with square ends and is designed
for operation with uniform current density i (Fig. 1). Fabry showed that the magnetic
field strength at the coil center H_0 can be expressed by:

$$H_0 = G\sqrt{\frac{P\lambda}{\rho a_1}} ,$$ (1)

where

P = ohmic loss power, in watts,

λ = "space factor" = $\dfrac{\text{conducting cross section}}{\text{total cross section including insulation}}$,

VOLUME FACTOR

$V = \pi (a_2^2 - a_1^2)2b = v\, a_1^3$

$v = 2\pi(a^2 - 1)\beta$

FABRY FACTOR

$H_0 = G\sqrt{\dfrac{P\lambda_1}{\rho_1 a_1}}$

$G = \dfrac{\sqrt{2\pi}}{5}\sqrt{\dfrac{\beta}{a^2-1}}\ \ell n\ \dfrac{a + \sqrt{a^2+\beta^2}}{1 + \sqrt{1+\beta^2}}$

$ = \dfrac{\sqrt{2\pi}}{5}\sqrt{\dfrac{\beta}{a^2-1}}\ \ell n\ \dfrac{a_2 + s_1}{a_1 + s_1}$

$i = $ CONSTANT

$a = \dfrac{a_2}{a_1}$; $\beta = \dfrac{b}{a_1}$

$i^+ = \rho^+ = \lambda^+ = 1$

COIL WITH RECTANGULAR
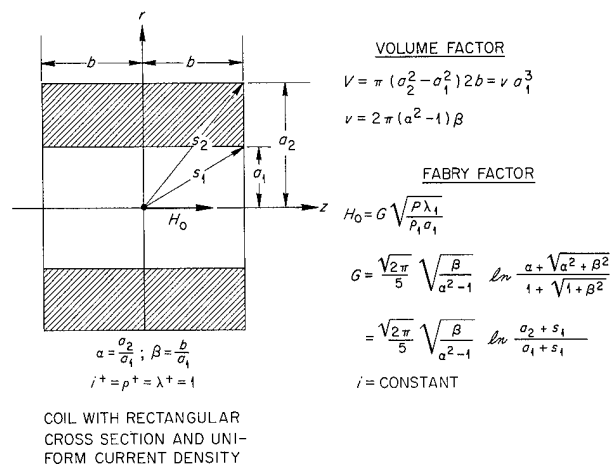CROSS SECTION AND UNI-
FORM CURRENT DENSITY

Figure 1.

ρ = resistivity of the coil conductor, in ohm-cm,

a_1 = inside radius of the coil, in cm.

H_0 is measured in oersteds, and B_0 has the same numerical value in gauss (no ferromagnetic material is supposed to be in the magnetic field). G is a dimensionless factor, the so-called "Fabry factor," which does not depend on the size but only on the shape of the coil.

By introducing the ratios:

$$\alpha = \frac{a_2}{a_1} ,$$

$$\beta = \frac{b}{a_1} ,$$

(2)

(see Fig. 1), G can be expressed as a function of α and β in an elementary way.[4a] It is convenient to represent G by a family of curves (see Fig. 2).[7] A "flat" maximum of $G = 0.179$ occurs for $\alpha = 3.09$, $\beta = 1.88$.

Fabry also used a "volume factor v," defined by:

$$v = \frac{\text{total coil volume}}{a_1^3} .$$

(3)

For instance, in the case of a cylindrical coil with square ends, $v = 2\pi(\alpha^2 - 1)\beta$. In Fig. 2, v = constant curves are shown by dashed lines. G_{max} corresponds to $v = 101$.
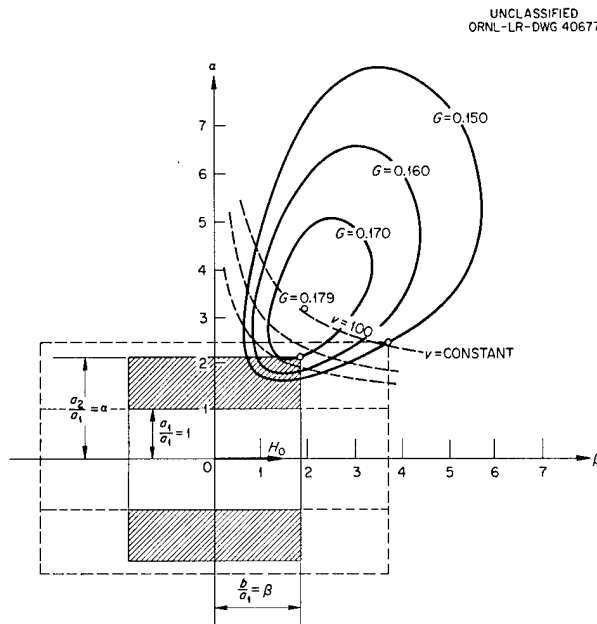
Fabry and Volume Factor Diagrams of Cylindrical Magnet Coils with Square Ends and Uniform Current Density.

Figure 2.

A somewhat smaller G can be achieved with a much smaller volume factor. For instance, the DCX coils are designed with the following data: [8]

$$\alpha = 1.53,$$
$$\beta = 0.618,$$
$$\nu = 5.21,$$
$$G = 0.128.$$

Fabry gave a table showing, for volume factors between 1 and 10^5, values of α and β for which G assumes maximal values.[9] The DCX coil is such a minimum-volume coil. It should be pointed out, however, that for many applications a minimum volume factor is not of special importance, since the whole field configuration produced by the magnet coil must be considered.

II. FABRY FACTOR OF A LAYER COIL

A coil shall be called a "layer coil" if it is wound in straight homogeneous layers; i.e., if current density i (measured in amp/cm^2), resistivity ρ, and space factor λ are functions only of the distance from the axis of r. If these quantities are functions of r and z, the coil will be called a "loop coil." Figure 3 shows a layer coil which is symmetrical to a plane through 0 perpendicular to the z-axis. In the following, a general expression for the Fabry factor G will be derived for a symmetric layer coil where the strength H_0 is referred to the coil center. A similar expression can be found for an asymmetric layer coil, and H_0 referred to any point at the axis (see Appendix II).

A cylindrical current sheet with a radius r, the length $2z$, and a thickness dr produces in the center the field strength: [10]

$$dH_0 = \frac{4\pi}{10} \lambda i \cos \phi \, dr = \frac{4\pi}{10} \lambda i \frac{z}{s} dr \ .$$

Therefore,

$$H_0 = \frac{4\pi}{10} \int_{r=a_1}^{a_2} \lambda i \frac{z}{s} dr \ . \tag{4}$$

The power dissipated in the infinitesimal current sheet is:

$$dP = \rho i^2 \, dV_{Cu} = \rho \lambda i^2 \, dV = 4\pi \rho \lambda i^2 rz \, dr \ .$$

Therefore,

$$P = 4\pi \int_{r=a_1}^{a_2} \rho \lambda i^2 rz \, dr \ . \tag{5}$$

Normalized (dimensionless) quantities, which are in general functions of r, are introduced in the following way:

$$i^+ = \frac{i}{i_1}; \quad \rho^+ = \frac{\rho}{\rho_1}; \quad \lambda^+ = \frac{\lambda}{\lambda_1} \ . \tag{6}$$

The subscript 1 refers to the innermost layer where $r = a_1$.

Equations 4 and 5 become:

$$H_0 = \frac{4\pi}{10} \lambda_1\, i_1 \int_{r=a_1}^{a_2} \lambda^+ i^+ \frac{z}{s}\, dr \ ,$$

(4a)

and

$$P = 4\pi \rho_1 \lambda_1\, i_1^2 \int_{r=a_1}^{a_2} \rho^+ \lambda^+ (i^+)^2 rz\, dr \ .$$

(5a)

Eliminating i_1 from these equations, we obtain[11]

$$H_0 = \sqrt{\frac{P\lambda_1}{\rho_1 a_1}} \ ,$$

(7a)

$$G = \frac{\sqrt{\pi a_1}}{5} \cdot \frac{\displaystyle\int_{r=a_1}^{a_2} \lambda^+ i^+ \frac{z}{s}\, dr}{\displaystyle\sqrt{\int_{r=a_1}^{a_2} \rho^+ \lambda^+ (i^+)^2 rz\, dr}} \quad \text{(layer coil)} \ .$$

(7b)

Equation 7b contains a_1; however, G is dimensionless and therefore independent of the magnitude of a_1.

III. FABRY FACTORS OF CYLINDRICAL COILS WITH TAPERED ENDS

As an example of the application of Eqs. 7a and 7b, a cylindrical coil with tapered ends (see Fig. 4), constant space factor ($\lambda^+ = 1$), and constant specific resistance

Optimum Current Density of a
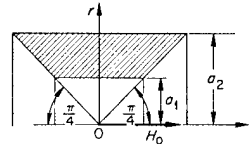Symmetrical Layer Coil
(Shape, ρ^+, and λ^+ Known).
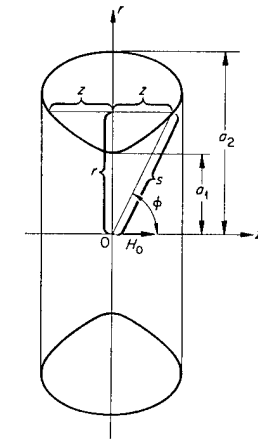
$\rho\, r s i$ = CONSTANT = ρ_1, a_1, s_1, i_1

$$G_{OPT} = \frac{1}{5}\sqrt{\pi a_1}\sqrt{\int_{r=a_1}^{a_2} \frac{\lambda^+}{\rho^+}\frac{z}{rs^2}\, dr}$$

$$J_{OPT} = \frac{a_1}{10 s_1} \frac{1}{G_{OPT}}$$

Fig. 3.

$\frac{a_2}{a_1} = a$ \quad ρ^+ AND λ^+ CONSTANT

ASSUMED: $i^+ = \left(\frac{a_1}{r}\right)^2$

$$G = 0.251\sqrt{\frac{a-1}{a}}$$

LIKEWISE SOLUTION OF CASE (c).

CYLINDRICAL COIL WITH
TAPERED ENDS

Fig. 4.

Figures 3 and 4.

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$(\rho^+ = 1)$ will be considered. The current density is also assumed to be constant, inversely proportional to the radius r, or inversely proportional to r^2.

a. $i^+ = 1$.

$$G = \frac{\sqrt{3\pi}}{5} \sqrt{\frac{k}{1 + k^2}} \frac{\alpha - 1}{\sqrt{\alpha^3 - 1}} . \tag{8}$$

The maximum value for G is 0.172, for an angle $\phi = 45°$ ($k = 1$), and $\alpha = 2.7$. This is case 4 of Bitter's paper.[6]

b. $i^+ = \dfrac{a_1}{r}$.

$$G = \frac{\sqrt{\pi}}{5} \sqrt{\frac{k}{1 + k^2}} \frac{\ln \alpha}{\sqrt{\alpha - 1}} . \tag{9}$$

The maximum value for G is 0.201, again for an angle $\phi = 45°$, but for $\alpha = 4.5$. This is Bitter's case 5.

c. $i^+ = \left(\dfrac{a}{r}\right)^2$.

$$G = \frac{\sqrt{\pi}}{5} \sqrt{\frac{k}{1 + k^2}} \sqrt{\frac{\alpha - 1}{\alpha}} . \tag{10}$$

The maximum value for G is 0.250, for an angle $\phi = 45°$, and α approaching infinity. It seems that this case has not been discussed previously; however, as will be shown, it deserves interest because this current distribution yields the maximum G for a cylindrical layer coil with tapered ends.

IV. OPTIMUM CURRENT DENSITY DISTRIBUTION OF A LAYER COIL

In the previous paragraph maximum values of G were considered which occur for special values of k and α, i.e., for special shapes of cylindrical coils with tapered ends. Another problem is to find for a coil of any definite shape a current density distribution $i(r)$ at which G becomes a maximum (optimum current density distribution). Special cases of this problem were discussed by Maxwell,[12] Fabry,[13] and Bitter.[6] Here a general law for the optimum current density distribution of a coil of known shape and a general expression for the Fabry factor (optimum Fabry factor) which results in this case are derived. For simplicity a symmetrical layer coil is supposed, and H_0 is referred to the coil center. The more general problem, to find optimum current density distribution and optimum Fabry factor for a layer coil of any shape, can be solved in a similar way. As shown in the Appendix I, a certain value of H_0 (Eq. 4) can be produced with a minimum power P (Eq. 5) if the following relation is satisfied.

$$\rho r s i = \rho_1 a_1 s_1 i_1 = \text{constant.} \tag{11a}$$

Thus

$$i^+ \frac{1}{\rho^+} \frac{a_1 s_1}{rs} . \tag{11b}$$

That is to say: *a desired magnetic field strength in the center of a symmetrical layer coil is achieved with minimum power when the product of specific resistance, radial distance of a layer from the axis, distance of the layer end-point from the coil center, and current density are constant.*

Substituting i^+ from Eq. 11b into Eq. 7b, we obtain the optimum Fabry factor:

$$G_{opt} = \frac{\sqrt{\pi a_1}}{5} \sqrt{\int_{r=a_1}^{a_2} \frac{\lambda^+}{\rho^+} \frac{z}{rs^2} \, dr} \quad \text{(layer coil)}. \tag{12}$$

Some examples will be discussed below.

V. CURRENT DENSITY FACTOR J OF A LAYER COIL

As shown above, the Fabry factor G is dimensionless and characteristic of a certain coil shape and the distribution of relative current density, space factor, and resistivity. The absolute value of H_0 is found by multiplying G by a scaling factor:

$$\sqrt{\frac{P\lambda_1}{\rho_1 a_1}},$$

(see Eq. 7a). A similar procedure may now be used to find the absolute value of the current density i_1. From Eq. 5a,

$$i_1 = \sqrt{\frac{P}{4\pi\rho_1\lambda_1 \int_{r=a_1}^{a_2} \rho^+\lambda^+(i^+)^2 rz \, dr}} . \tag{13}$$

This equation can be written as:

$$i_1 = J \sqrt{\frac{P}{\rho_1\lambda_1 a_1^3}} ; \tag{14a}$$

$$J = \frac{1}{2} \sqrt{\frac{a_1^3}{\pi}} \sqrt{\frac{1}{\int_{r=a_1}^{a_2} \rho^+\lambda^+(i^+)^2 rz \, dr}} . \tag{14b}$$

The dimensionless factor J shall be called the "current density factor." The absolute value of i_1 is found by multiplying J by a scaling factor:

$$\sqrt{\frac{P}{\rho_1\lambda_1 a_1^3}} .$$

The analogy to the Fabry factor is obvious.

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Two special cases might be considered first. For a coil of any shape with constant resistivity, space factor, and current density, the current density factor becomes simply (see Eq. 3):

$$J = \frac{1}{\sqrt{v}} \quad (\rho^+ = \lambda^+ = i^+ = 1) \quad . \tag{15}$$

In this special case the current density factor is the reciprocal of the square root of the volume factor.

For a symmetrical layer coil of any shape, operated with optimum current distribution (Eq. 11b), the current density factor (Eq. 14b) becomes:

$$J_{opt} = \frac{1}{2s_1} \sqrt{\frac{a_1}{\pi}} \frac{1}{\sqrt{\int_{r=a_1}^{a_2} \frac{\lambda^+}{\rho^+} \frac{z}{rs^2} dr}} \quad . \tag{16}$$

Or, considering Eq. 12:

$$J_{opt} = \frac{a_1}{10s_1} \frac{1}{G_{opt}} \quad \text{(layer coil)} \quad . \tag{17}$$

When the Fabry factor of a symmetrical coil with optimum current distribution is known, the absolute values of the current densities can be found easily by Eqs. 17, 14a, and 11b. For a coil section similar to that shown in Fig. 3, $a_1 = s_1$, and therefore

$$J_{opt} = \frac{1}{10} \frac{1}{G_{opt}} \quad \text{(layer coil)}, \ a_1 = s_1 \quad . \tag{17a}$$

VI. EXAMPLES OF LAYER COILS WITH OPTIMUM CURRENT DENSITY DISTRIBUTION

a. **Cylindrical Coil with Square Ends and Optimum Radial Current Distribution,** $\rho^+ = \lambda^+ = 1$ **(see Fig. 1).**[4b,6]

Equation 11b becomes in this case:

$$i^+ = \frac{a_1 \sqrt{a_1^2 + b^2}}{r \sqrt{b^2 + r^2}} , \tag{18}$$

and Eq. 12 yields:

$$G_{opt} = \frac{1}{10} \sqrt{\frac{2\pi}{\beta} \ln \frac{\alpha^2(1 + \beta^2)}{\alpha^2 + \beta^2}} \quad . \tag{19}$$

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For a coil with $\alpha = 3$, $\beta = 2$, the Fabry factor is $G_{opt} = 0.197$. From Eq. 17 the current density factor is:

$$J_{opt} = \frac{1}{10\sqrt{5}}\frac{1}{0.197} = 0.226 \quad .$$

Equation 11b yields for the current density at $r = a_2 = 3a_1$,

$$i_2 = \frac{a_1 s_1}{a_2 s_2} i_1 = \frac{\sqrt{5}}{3\sqrt{13}} i_1 = 0.206\, i_1 \quad .$$

It is interesting to compare these data with those for a cylindrical coil with square ends and constant current density and with the same a_1, α, β, ρ, and λ. The field strength at the coil center in both cases is the same. Then the power ratio is (see Eq. 1):

$$\frac{P}{P'} = \left(\frac{G'}{G}\right)^2 = \left(\frac{0.179}{0.197}\right)^2 = 0.825 \quad .$$

That is to say, 17.5% of the power can be saved by optimizing the current density distribution. From Eq. 15:

$$J' = \frac{1}{\sqrt{v}} = 0.100 \quad .$$

Therefore (see Eq. 14a),

$$\frac{i_1}{i_1'} = \frac{J\sqrt{P}}{J'\sqrt{P'}} = \frac{JG'}{J'G} = \frac{0.226 \times 0.179}{0.100 \times 0.197} = 2.05 \quad .$$

The current density in the optimized coil at $r = a_1$ is $2.05\, i_1'$, and decreases to $2.05\, i_1' \times 0.206 = 0.424\, i_1'$ at $r = a_2$.

b. **Cylindrical Coil with Tapered Ends and Optimum Radial Current Distribution,**
    $\rho^+ = \lambda^+ = 1$ (see **Fig. 4).**

Equation 11b becomes in this case:

$$i^+ = \left(\frac{a_1}{r}\right)^2 \quad . \tag{20}$$

A cylindrical coil with tapered ends (resistivity and space factor constant) has a maximum Fabry factor when the current density is inversely proportional to the square of the distance from the axis. This is exactly the previously considered third example of Section III. For $k = 1$ and $\alpha = 3$, Eq. 10 yields $G_{opt} = 0.204$. From Eq. 17, $J_{opt} = 0.346$.

From Eq. 8 it follows that for a cylindrical coil with tapered ends, with $k = 1$ and $\alpha = 3$ but with constant current density, $G' = 0.170$. With H_0 the same in each coil,

$$\frac{P}{P'} = \left(\frac{0.170}{0.204}\right)^2 = 0.695 \quad ;$$

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that is, 30.5% power can be saved by optimizing the current density distribution. From Eq. 15,

$$J' = \frac{1}{\sqrt{\dfrac{4\pi}{3}(\alpha^3 - 1)}} = \frac{1}{\sqrt{108.9}} = 0.0958 \ .$$

Therefore,

$$\frac{i_1}{i_1'} = \frac{0.346 \times 0.170}{0.0458 \times 0.204} = 3.01 \ .$$

The current density in the optimized coil at $r = a_1$ is $3.05\,i_1'$ and decreases to $\frac{1}{9}$ of this value, i.e., to $0.334\,i_1'$ at $r = a_2$. Whether the absolute value of the maximum current density is acceptable or not depends on the numerical value of the scaling factor,

$$\sqrt{\frac{P}{\rho_1 \lambda_1 a_1^3}} \ .$$

VII. SHAPE OPTIMIZATION OF LAYER COILS

Section IV dealt with finding the optimum current density distribution of a layer coil. Another problem is that of the optimum shape of a coil (i.e., where G is a maximum) when the current density distribution $i(r)$ is known. For simplicity, only symmetrical layer coils will be considered. The case of an unsymmetrical layer coil is discussed in Appendix II.

It can be shown (see Appendix I) that the optimization condition in this case is:

$$\frac{\rho s^3 i}{r} = \frac{\rho(r^2 + z^2)^{3/2} i}{r} \ . \tag{21}$$

We consider the following special cases:

a. $\rho^+ = i^+ = 1$ (Constant Current Density)

Using the coordinates s and ϕ (see Fig. 3), we obtain for the contour curve of the coil the equation:

$$\frac{s^2}{\sin \phi} = \text{constant.} \tag{22}$$

This case was considered by Fabry.[4b] He found $G_{max} = 0.18$.

b. $\rho^+ = 1$, $i^+ = a_1/r$ (Current Density Inversely Proportional to the Distance from the Axis)

The equation of the contour curve is:

$$\frac{s}{\sin^2 \phi} = \text{constant.} \tag{23}$$

The Fabry factor is again near 0.18.

c. $\rho^+ = 1$, $i^+ = (a_1/r)^2$ (Current Density Inversely Proportional to the Square of the Distance from the Axis)

In this case:

$$\phi = \text{constant} = \frac{\pi}{4} \ . \tag{24}$$

This layer coil is a coil with tapered ends, as considered in sections IIIc and VIb. It can be shown that this type of a layer coil represents the solution of the problem of finding the optimum Fabry factor when both shape and current density distribution are optimized simultaneously. However, in this case the maximum current density is relatively very high, and this fact, combined with the difficulties of building such coils, seems to limit their practical application.

VIII. LOOP COILS

As mentioned previously, a loop coil, in contrast to a layer coil, is one in which the quantities i, r, ρ, and λ are functions of both coordinates r and z. Loop coils have little practical importance (although a slightly higher G is attainable) since their design is difficult. However, some basic theorems concerning loop coils and some numerical examples may be of interest.

Assume that a loop has an infinitesimally small cross sectional area dA, in the z-r plane, and that its location and radius is determined by the coordinates a and r (distance from the origin $s = \sqrt{z^2 + r^2}$); then the magnetic field strength produced at the origin is:[10]

$$H_0 = \frac{2\pi}{10} \int_A \lambda \frac{r^2}{s^3} i \, dA \ . \tag{25}$$

When normalized with respect to an appropriately selected reference loop (indicated by the subscript 1), this equation becomes:

$$H_0 = \frac{2\pi}{10} \lambda_1 i_1 \int_A \lambda^+ \frac{r^2}{s^3} i^+ \, dA \ . \tag{25a}$$

The power dissipated is:

$$P = 2\pi \int_A \rho \lambda r i^2 \, dA \ , \tag{26}$$

or, normalized,

$$P = 2\pi \rho_1 \lambda_1 i_1^2 \int_A \rho^+ \lambda^+ r (i^+)^2 \, dA \ . \tag{26a}$$

Corresponding to Eqs. 7a and 7b we obtain now for a loop coil:

$$H_0 = G \sqrt{\frac{P \lambda_1}{\rho_1 a_1}} \ , \tag{27a}$$

$$G = \frac{\sqrt{2\pi a_1}}{10} \frac{\int_A \lambda^+ \frac{r^2}{s^3} i^+ \, dA}{\sqrt{\int_A \rho^+ \lambda^+ r(i^+)^2 \, dA}} \quad \text{(loop coil)} \quad .$$ (27b)

For the current density factor we obtain (compare Eqs. 14a and 14b):

$$i_1 = J \sqrt{\frac{P}{\rho_1 \lambda_1 a_1^3}} \quad ,$$ (28a)

$$J = \sqrt{\frac{a_1^3}{2\pi}} \frac{1}{\sqrt{\int_A \rho^+ \lambda^+ r(i^+)^2 \, dA}} \quad \text{(loop coil)} \quad .$$ (28b)

J. C. Maxwell solved the problem of designing a galvanometer in which a given small electromotive force would produce the greatest possible deflection.[12] In a more general way we can ask about the optimum current density distribution of a loop coil of any shape, for which ρ and λ are known functions of z and r. Reasoning along the lines which led to finding the optimum current density distribution of a layer coil (see Appendix I) leads to the following optimization condition of a loop coil (compare with Eqs. 11a and 11b):

$$\frac{\rho s^3 i}{r} = \frac{\rho_1 s_1^3 i_1}{r_1} = \text{constant} \quad ,$$ (29a)

$$i^+ = \frac{1}{\rho^+} \frac{s_1^3 r}{r_1 s^3} = \frac{1}{\rho^+} \frac{s_1^2}{\sin \phi_1} \frac{\sin \phi}{s^2} \quad .$$ (29b)

Substituting this value in Eqs. 27b and 28b we obtain:

$$G_{opt} = \frac{\sqrt{2\pi a_1}}{10} \sqrt{\int_A \frac{\lambda^+}{\rho^+} \left(\frac{\sin \phi}{s}\right)^3 dA} \quad ;$$ (31)

$$J_{opt} = \frac{\sin^3 \phi_1}{10 \, G_{opt}} \quad \text{(loop coil)} \quad .$$ (32)

Equation 29b shows that the current density distribution does not depend on the shape of the cross sectional area, A, of the coil since the coordinates of the contour of the coil cross section do not appear in this equation. A family of curves can be drawn to represent the i = constant lines which yield a maximum G for any shape of coil cross section (see Fig. 5a). From Eq. 31 it follows that G_{opt} assumes the highest values if the coil cross section A extends from points as close as possible to its axis of symmetry. This is illustrated by the following examples of loop coils with optimized current density distribution.

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a. **Spherical Coil (Inside Radius a_1, Outside Radius a_2, see Fig. 5b)**

A spherical coil is difficult to construct, since it must be built in two halves in order to provide access to the inner empty sphere. In this case:

$$G_{opt} = \frac{1}{5}\sqrt{\frac{2\pi}{3}}\sqrt{\frac{\alpha - 1}{\alpha}} \quad . \tag{33}$$

For very large α the Fabry factor approaches the value 0.289.[4b]

b. **Long Cylindrical Solenoid (Inside Radius a_1, Outside Radius a_2)**

$$G_{opt} = \frac{\pi\sqrt{3}}{20}\sqrt{\frac{\alpha - 1}{\alpha}} \quad . \tag{34}$$

For very large α the Fabry factor approaches the value 0.272.[4b,6]

c. **Cylindrical Coil with Tapered Ends (Fig. 4)**

$$G_{opt} = \frac{\sqrt{2\pi}}{10}\sqrt{\frac{3}{4}\left(\frac{\pi}{2} - \phi\right) + \frac{1}{2}\sin 2\phi - \frac{1}{16}\sin 4\phi}\sqrt{\frac{\alpha - 1}{\alpha}} \quad . \tag{35}$$

For $\phi = \pi/4$:

$$G_{opt} = 0.262\sqrt{\frac{\alpha - 1}{\alpha}} \quad . \tag{35a}$$

This yields $G_{opt} = 0.213$ for $\alpha = 3$.

As shown in section VIb a cylindrical layer coil of the same shape optimized in respect to current density distribution has a Fabry factor $G_{opt} = 0.204$, which is only slightly smaller. The current density factor J_{opt}, however, increases noticeably from 0.346 for the layer coil to 0.470 for the optimized loop coil. This shows clearly the merely academic interest of this case.

Finally the following theorem might be derived. Consider a set of n loop coils of any cross sections (Fig. 6). The optimized Fabry factor of the coil k is (Eq. 31):
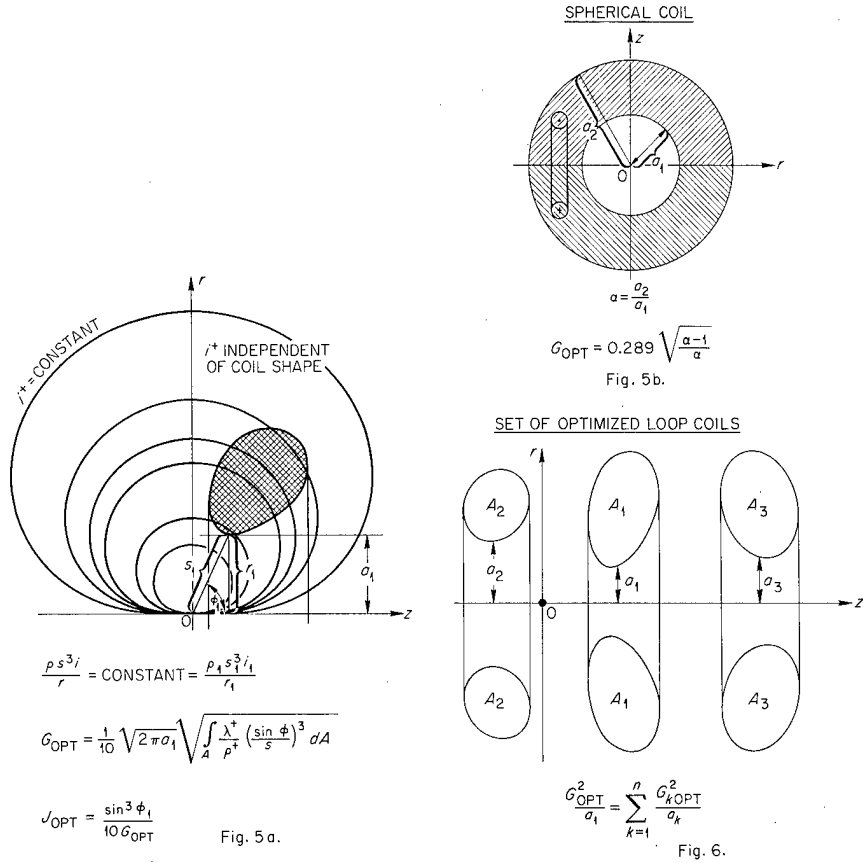
$$G_{k\,opt} = \frac{\sqrt{2\pi a_k}}{10}\sqrt{\iint_{A_k}\frac{\lambda^+}{\rho^+}\left(\frac{\sin\phi}{s}\right)^3 dA} \quad . \tag{36}$$

The optimized Fabry factor of the whole loop arrangement is:

$$G_{opt} = \frac{\sqrt{2\pi a_1}}{10}\sqrt{\iint_A\frac{\lambda^+}{\rho^+}\left(\frac{\sin\phi}{s}\right)^3 dA} \quad . \tag{37a}$$

The integration has to be performed over the total cross sectional area A, i.e., over all the incremental areas A_k. Expressed in somewhat different notation:

$$G_{opt} = \frac{\sqrt{2\pi a_1}}{10}\sqrt{\sum_{k=1}^{n}\int_{A_k}\frac{\lambda^+}{\rho^+}\left(\frac{\sin\phi}{s}\right)^3 dA_k} \quad . \tag{37b}$$

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SPHERICAL COIL

$$a = \frac{a_2}{a_1}$$

$$G_{OPT} = 0.289 \sqrt{\frac{a-1}{a}}$$

Fig. 5b.

SET OF OPTIMIZED LOOP COILS

$$\frac{G_{OPT}^2}{a_1} = \sum_{k=1}^{n} \frac{G_{k\,OPT}^2}{a_k}$$

Fig. 6.

i^+ = CONSTANT

i^+ INDEPENDENT OF COIL SHAPE

$$\frac{\rho\, s^3 i}{r} = \text{CONSTANT} = \frac{\rho_1 s_1^3 i_1}{r_1}$$

$$G_{OPT} = \frac{1}{10} \sqrt{2\pi a_1} \sqrt{\int_A \frac{\lambda^+}{\rho^+} \left(\frac{\sin\phi}{s}\right)^3 dA}$$

$$J_{OPT} = \frac{\sin^3\phi_1}{10\,G_{OPT}}$$

Fig. 5a.

Optimum Current Density Distribution of a Loop Coil
(ρ^+ and λ^+ Known).

Figures 5a, 5b, and 6.

Therefore,

$$\frac{G_{opt}^2}{a_1} = \sum_{k=1}^{n} \frac{G_{k\,opt}^2}{a_k} \,. \tag{38}$$

The Fabry factor is increased by the addition of more coils to the set. Its highest value is achieved when the whole available space is filled with windings. This is in contrast to layer coils. For these the Fabry factor decreases when a certain optimum winding space is exceeded.

CONCLUSION

1. General expressions for the Fabry factor of a "layer coil" (for which coil variables depend on the distance r from the axis only), and for a "loop coil" (coil variables depend on r and on axial coordinate z) have been derived (Eq. 7b and 27b). These general forms are convenient for the computation of the Fabry factor of a coil of any shape and any distribution of current density, resistivity, and space factor.

2. In order to produce a desired field strength H_0 at any point on the axis, with the minimum power P, the following kinds of optimization are possible:

    a. The coil shape and the distribution of ρ and λ are known; the problem is to find the optimum current density distribution. For symmetrical loop coils and layer coils with variable ρ and λ general solutions have been found (Eq. 11b and 29b). The following result is of particular interest: A desired magnetic field strength in the center of a symmetrical layer coil is achieved with the minimum power when the product of resistivity ρ, radial distance r of a layer from the axis, distance of the layer end point from the coil center s, and current density i is constant.

    b. The current density distribution of a layer coil is any known function $i = i(r)$, likewise the distributions of ρ and λ are known; find the shape of the optimum coil. A general solution has been found (Eq. 21). Of particular interest is the current density distribution:

$$i = \left(\frac{a_1}{r}\right)^2 i_1 \;.$$

In this case the optimum coil is cylindrical with tapered ends.

    c. Simultaneously optimizing current density distribution and coil shape likewise leads to the cylindrical coil with tapered ends and current density inversely proportional to the square of the distance from the axis.

3. For any magnet coil, the current density, i_1, in the innermost layer (or in the loop with smallest radius) can be represented by the product of a dimensionless factor, the "current density factor" J, and a scaling factor as follows:

$$i_1 = J \sqrt{\frac{P}{\rho_1 \lambda_1 a_1^3}} \;. \tag{14a}$$

The analogy to the Fabry factor is obvious. General expressions for the current density factors of layer and loop coils have been derived (Eq. 14b and 28b). For a coil of any shape with constant resistivity, space factor, and current density, the current density factor J is the reciprocal value of the square root of the volume factor (Eq. 15).

4. For coils with optimum current density distribution, simple general expressions for the Fabry factor G_{opt} have been derived (Eq. 12 and 21). The current density factor of a layer coil becomes in this case:

$$J_{opt} = \frac{a_1}{10\,s_1} \frac{1}{G_{opt}} \quad \text{(layer coil)} \;, \tag{17}$$

and that of a loop coil:

$$J_{opt} = \frac{\sin^3 \phi_1}{10} \frac{1}{G_{opt}} \quad \text{(loop coil)} \;. \tag{32}$$

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5. The Fabry factor, optimized with respect to current density, of n loop coils, each having the optimized Fabry factors $G_{k\,opt}$, is:

$$\frac{G^2_{opt}}{a_1} = \sum_{k=1}^{n} \frac{G^2_{k\,opt}}{a_k} \ . \tag{38}$$

Numerical examples illustrate the basic concepts and theorems mentioned above.

APPENDIX I

Bitter[6] showed how to find the optimum current density distribution and the corresponding values of the Fabry factor of coils of known shape. He did this by the method of indeterminate multipliers, and the same mathematical method shall be applied here to the optimization of the current density $i(r)$ of a layer coil of any symmetrical shape and with any $\rho(r)$ and $\lambda(r)$ distribution.

From Eqs. 4 and 5,

$$F[i(r)] = H_0 + \mu P = \frac{4\pi}{10} \int_{r=a_1}^{a_2} \lambda i \frac{z}{s}\,dr + \mu\,4\pi \int_{r=a_1}^{a_2} \rho \lambda i^2 r\,z\,dr \ , \tag{a}$$

μ being an arbitrary constant.

$$\delta F[i(r)] = 0 \tag{b}$$

is satisfied if:

$$\frac{4\pi}{10} \lambda \frac{z}{s} + \mu\,8\pi \rho \lambda i r z = 0 \ . \tag{c}$$

Therefore:

$$\rho r s i = -\frac{1}{20\mu} = \text{constant} = \rho_1 a_1 s_1 i_1 \ . \tag{d}$$

This is Eq. 11a. Note that $\lambda(r)$ does not occur in this optimization condition for $i(r)$. The space factor $\lambda(r)$, of course, influences the Fabry factor, as shown in Eq. 12. A similar calculation can be used to prove Eq. 21.

APPENDIX II

Equations for Unsymmetrical Layer Coils

Figure 7 shows a layer coil with the z-axis as the axis of rotational symmetry. The coil, however, is not symmetrical to any plane perpendicular to the z-axis (unsymmetrical layer coil). Referring to Fig. 7, the abbreviation,

$$\cos \phi_{11} - \cos \phi_1 = \psi \ , \tag{e}$$

is used. The magnetic field strength at the point 0 assumes the form

$$H_0 = \frac{\pi}{5} \int_{r=a_1}^{a_2} \lambda \psi i\,dr \ . \tag{f}$$
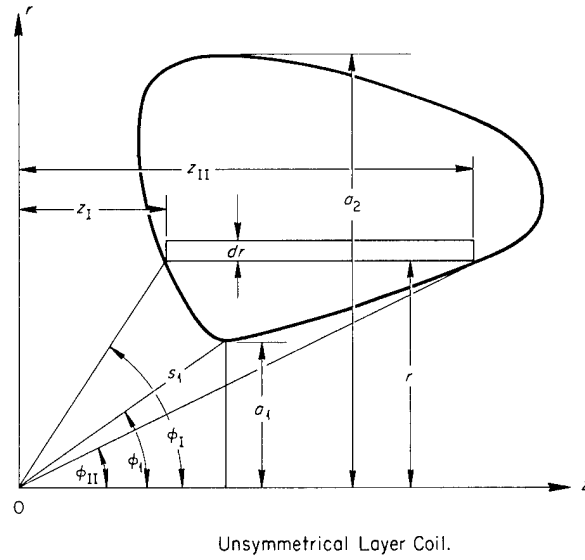
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Unsymmetrical Layer Coil.

Figure 7.

This can be written in normalized form:

$$H_0 = \frac{\pi}{5} \lambda_1 i_1 \int_{r=a_1}^{a_2} \lambda^+ \psi i^+ dr \ . \tag{g}$$

The ohmic power loss becomes

$$P = 2\pi \int_{r=a_1}^{a_2} \rho \lambda r (z_{II} - z_1) i^2 \, dr$$

$$= 2\pi \rho_1 \lambda_1 i_1^2 \int_{r=a_1}^{a_2} \rho^+ \lambda^+ r (z_{II} - z_1)(i^+)^2 \, dr \ . \tag{h}$$

From these equations it follows that the Fabry factor

$$G = \frac{1}{5} \sqrt{\frac{\pi a_1}{2}} \frac{\int_{r=a_1}^{a_2} \lambda^+ \psi i^+ \, dr}{\sqrt{\int_{r=a_1}^{a_2} \rho^+ \lambda^+ r (z_{II} - z_1)(i^+)^2 \, dr}} \ , \tag{i}$$

and the current density factor

$$J = \sqrt{\frac{a_1^3}{2\pi}} \frac{1}{\sqrt{\int_{r=a_1}^{a_2} \rho^+ \lambda^+ r (z_{II} - z_1)(i^+)^2 \, dr}} \ . \tag{j}$$

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The current density distribution for which G is a maximum can be found by the method discussed in Appendix I. The result is

$$\frac{\rho(z_{II} - z_I)r}{\psi} i = \text{constant} .$$

(k)

If we consider a coil cross section similar to that shown in Fig. 7, Eq. (k) applied to the point $r = a_1$ assumes the form $0/0$, which evaluates to

$$\frac{\rho(z_{II} - z_I)r}{\psi} i = \frac{\rho_1 a_1^2}{\sin^3 \phi_1} i_1 ,$$

(l)

or

$$i^+ = \frac{\psi}{\sin^3 \phi_1} \frac{a_1^2}{(z_{II} - z_I)r\rho^+} .$$

(m)

This value substituted in Eq. (i) and (j) leads to

$$G_{opt} = \frac{1}{5} \sqrt{\frac{\pi a_1}{2}} \sqrt{\int_{a_1}^{a_2} \frac{\lambda^+ \psi^2}{\rho^+} \frac{dr}{(z_{II} - z_I)r}} ;$$

(n)

$$J_{opt} = \frac{\sin^3 \phi_1}{10} \frac{1}{G_{opt}} .$$

(o)

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